

Jones basic construction on field algebras of G -spin models ^{*}

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Abstract: Let G be a finite group. Starting from the field algebra \mathcal{F} of G -spin models, one can construct the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$ such that it coincides with the C^* -basic construction for the field algebra \mathcal{F} and the $D(G)$ -invariant subalgebra of \mathcal{F} , where $D(G)$ is the quantum double of G . Under the natural $\widehat{D(G)}$ -module action on $\mathcal{F} \rtimes D(G)$, the iterated crossed product C^* -algebra can be obtained, which is C^* -isomorphic to the C^* -basic construction for $\mathcal{F} \rtimes D(G)$ and the field algebra \mathcal{F} . Furthermore, one can show that the iterated crossed product C^* -algebra is a new field algebra and give the concrete structure with the order and disorder operators.

Keywords: G -spin models, C^* -basic construction, field algebras, dual action

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1 Introduction

In [12], V.F.R.Jones introduced the notion of indices for inclusions of II_1 factors, which opened a completely new aspect of operator algebras involving various other fields of mathematics and physics, including topology, quantum physics, dynamical systems, noncommutative geometry, etc. He established a method to study the inner structures or the outer structures of given II_1 -factors, preserving certain quantity, called the Jones index. And he enlarged the pure algebraic Galois theory to operator algebra, and this has a significant impact on mathematics and applied mathematics. The interesting fact is that not only his theory but also the techniques he used in his theory are important mathematically and applicable to other fields [13]. Subsequently, there are lots of attempts to extend the original Jones theory by various mathematicians. For instance, M.Pimsner and S.Popa in [19] introduced the notion of the probabilistic index for a conditional expectation, which is the best constant of the so-called Pimsner-Popa inequality. In [15], H.Kosaki discussed another way to define index for a normal semifinite faithful conditional expectation of an arbitrary factor onto a subfactor exploiting spatial theory of Connes and the theory of operatorvalued weights.

Though the probabilistic index works perfectly for analytic purposes even in the case of C^* -algebras (see, e.g. [20]), it is not always suitable for algebraic operations such as the basic construction in the C^* -case. Inspired by the Pimsner-Popa basis in the sense of [19], Kosaki's index

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formula in the sense of [15], and the Casimir elements for semi-simple Lie algebras, Y. Watatani [24] proposed to assume existence of a quasi-basis for a conditional expectation, a generalization of the Pimsner-Popa basis in the von Neumann algebra case, to analyze inclusions of C^* -algebras. With a quasi-basis, Watatani successfully introduced a C^* -version of basic construction, which is closely related to K -theory of C^* -algebras [14, 24]. Roughly speaking, the (original or extended) Jones index theory is the study of certain elements measuring the maximal number of disjoint copies of topological $*$ -subalgebra in a given topological $*$ -algebra.

On the other hand, quantum chains considered as models of 1 + 1-dimensional quantum field theory exhibit many interesting features which include the emergence of braid group statistics and quantum symmetry. In particular, one-dimensional G -spin models, as a testing ground for applications of quantum group symmetries, have an order-disorder type of quantum symmetry given by the double $D(G)$ of a finite group G , defined by Drinfel'd in the context of finding solutions to the quantum Yang-Baxter equation arising from statistical mechanics [27], which generalizes the $Z(2) \times Z(2)$ symmetry of the lattice Ising model. In [22], the implementation of the Doplicher-Haag-Roberts theory of superselection sectors [6, 7, 8, 9] to G -spin models has been carried out. In this approach the symmetries can be reflected as the $D(G)$ -invariant subalgebra \mathcal{A} of the field algebra \mathcal{F} in G -spin models, called the observable algebra.

The paper studies the Jones basic construction on field algebras of G -spin models, and is organized as follows.

In Section 2, we collect the necessary definitions and facts about G -spin models, such as the quantum double $D(G)$, the field algebra \mathcal{F} and the Hopf action of the symmetry algebra $D(G)$ on \mathcal{F} , and then we also give a brief description of the C^* -basic construction for C^* -algebras.

In Section 3, under the conditional expectation E from the field algebra \mathcal{F} onto the $D(G)$ -invariant subalgebra \mathcal{A} , we can construct the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$, and then prove that this algebra is C^* -isomorphic to the C^* -algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

In Section 4, we show that there is a natural $\widehat{D(G)}$ -module algebra structure on $\mathcal{F} \rtimes D(G)$, and there exists a conditional expectation $E_2: \mathcal{F} \rtimes D(G) \rightarrow \mathcal{F}$ such that E_2 is consistent with the dual conditional expectation of $E: \mathcal{F} \rightarrow \mathcal{A}$ in Proposition 2.1. Moreover, we give the quasi-basis for the dual conditional expectation of E .

The main result of Section 5 is that the C^* -algebra constructed from the C^* -basic construction $\langle \mathcal{F} \rtimes D(G), e_2 \rangle$ for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$ is C^* -isomorphic to the iterated crossed product C^* -algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$, which is canonically isomorphic to $M_{|G|^2}(\mathcal{F})$ by Takai duality [23]. Then we give the concrete description of the new field algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ by means of the order and disorder operators.

All the algebras in this paper will be unital associative algebras over the complex field \mathbb{C} . The unadorned tensor product \otimes will stand for the usual tensor product over \mathbb{C} . For general results on Hopf algebras please refer to the books of Abe [1] and Sweedler [21]. We shall follow their notations, such as S , Δ , ε for the antipode, the comultiplication and the counit, respectively. Also

we shall use the so-called ‘‘Sweedler-type notation’’ for the image of Δ . That is

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

2 G -spin models and the C^* -basic construction

We first recall the main features of G -spin models, considered in the C^* -algebraic framework for quantum lattice systems, and then give the C^* -basic construction for C^* -algebras in G -spin models.

2.1 Definitions and preliminary results

Assume that G is a finite group with a unit u . The G -valued spin configuration on the two-dimensional square lattices is the map $\sigma: \mathbb{Z}^2 \rightarrow G$ with Euclidean action functional:

$$S(\sigma) = \sum_{(x,y)} f(\sigma_x^{-1} \sigma_y),$$

in which the summation runs over the nearest neighbor pairs in \mathbb{Z}^2 and $f: G \rightarrow \mathbb{R}$ is a function of the positive type. This kind of classical statistical systems or the corresponding quantum field theories are called G -spin models [10, 11, 25]. And such models provide the simplest examples of lattice field theories exhibiting quantum symmetry. In general, G -spin models with an Abelian group G are known to have a symmetry group $G \times \tilde{G}$, where \tilde{G} is the group of characters of G . If G is non-Abelian, the models have a symmetry of a quantum double $D(G)$ [5, 17], which is defined as follows.

Definition 2.1. Let $C(G)$ be the algebra of complex valued functions on G and consider the adjoint action of G on $C(G)$ according to $\alpha_g: f \mapsto f \circ \text{Ad}(g^{-1})$. The quantum double $D(G)$ is defined as the crossed product $D(G) = C(G) \rtimes_{\alpha} G$ of $C(G)$ by this action. In terms of generators $D(G)$ is the algebra generated by elements U_g and V_h ($g, h \in G$), with the relations

$$\begin{aligned} U_g U_h &= \delta_{g,h} U_g \\ V_g V_h &= V_{gh} \\ V_h U_g &= U_{hgh^{-1}} V_h, \end{aligned}$$

and the identification $\sum_{g \in G} U_g = V_u = 1$, where $\delta_{g,h} = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$

It is easy to see that $D(G)$ is of finite dimension, where as a convenient basis one may choose $U_g V_h$, $g, h \in G$, multiplying according to $U_{g_1} V_{h_1} U_{g_2} V_{h_2} = \delta_{g_1 h_1, h_1 g_2} U_{g_1} V_{h_1 h_2}$.

Here and from now on, by (g, h) we always denote the element $U_g V_h$ for notational convenience.

Also, the structure maps are given by

$$\begin{aligned} \Delta(g, h) &= \sum_{t \in H} (t, h) \otimes (t^{-1} g, h), & (\text{coproduct}) \\ \varepsilon(g, h) &= \delta_{g,u}, & (\text{counit}) \\ S(g, h) &= (h^{-1} g^{-1} h, h^{-1}), & (\text{antipode}) \end{aligned}$$

on the linear basis $\{(g, h), g, h \in G\}$ and are extended in $D(G)$ by linearity. One can prove that $D(G)$ is a Hopf algebra, with a unique element $E = \frac{1}{|G|} \sum_{g \in G} (u, g)$, called an integral element, satisfying for any $a \in D(G)$,

$$aE = Ea = \varepsilon(a)E.$$

Moreover, with the definition

$$(g, h)^* = (h^{-1}gh, h^{-1}),$$

and the appropriate extension, $D(G)$ is a semisimple $*$ -algebra of finite dimension [21], which implies that $D(G)$ becomes a Hopf C^* -algebra.

As in the traditional case, one can define the local quantum field algebra as follows.

Definition 2.2. The local field algebra of a G -spin model \mathcal{F}_{loc} is an associative algebra with a unit I generated by $\{\delta_g(x), \rho_h(l) : g, h \in G, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}\}$ subject to

$$\begin{aligned} \sum_{g \in G} \delta_g(x) &= I = \rho_u(l), \\ \delta_{g_1}(x) \delta_{g_2}(x) &= \delta_{g_1, g_2} \delta_{g_1}(x), \\ \rho_{h_1}(l) \rho_{h_2}(l) &= \rho_{h_1 h_2}(l), \\ \delta_{g_1}(x) \delta_{g_2}(x') &= \delta_{g_2}(x') \delta_{g_1}(x), \\ \rho_h(l) \delta_g(x) &= \begin{cases} \delta_{hg}(x) \rho_h(l) & \text{if } l < x, \\ \delta_g(x) \rho_h(l) & \text{if } l > x, \end{cases} \\ \rho_{h_1}(l) \rho_{h_2}(l') &= \begin{cases} \rho_{h_2}(l') \rho_{h_2^{-1} h_1 h_2}(l) & \text{if } l > l', \\ \rho_{h_1 h_2 h_1^{-1}}(l') \rho_{h_1}(l) & \text{if } l < l', \end{cases} \end{aligned}$$

for $x, x' \in \mathbb{Z}$, $l, l' \in \mathbb{Z} + \frac{1}{2}$ and $h_1, h_2, g_1, g_2 \in G$.

The $*$ -operation is defined on the generators as $\delta_g^*(x) = \delta_g(x)$, $\rho_h^*(l) = \rho_{h^{-1}}(l)$ and can be extended to an involution on \mathcal{F}_{loc} . In this way, \mathcal{F}_{loc} becomes a unital $*$ -algebra.

For any finite subset $\Lambda \subseteq \frac{1}{2}\mathbb{Z}$, let $\mathcal{F}(\Lambda)$ be the $*$ -subalgebra of \mathcal{F}_{loc} generated by

$$\left\{ \delta_g(x), \rho_h(l) : g, h \in G, x \in \Lambda \cap \mathbb{Z}, l \in \Lambda \cap (\mathbb{Z} + \frac{1}{2}) \right\}.$$

In particular, we consider an increasing sequence of intervals $\Lambda_n, n \in \mathbb{N}$, where

$$\begin{aligned} \Lambda_{2n} &= \{s \in \frac{1}{2}\mathbb{Z} : -n + \frac{1}{2} \leq s \leq n\}, \\ \Lambda_{2n+1} &= \{s \in \frac{1}{2}\mathbb{Z} : -n - \frac{1}{2} \leq s \leq n\}. \end{aligned}$$

In [22], the authors have shown that $\mathcal{F}(\Lambda_n), n \in \mathbb{N}$ are full matrix algebras, which can be identified with $M_{|G|^n}$. Moreover, under the induced norm, $\mathcal{F}(\Lambda_n)$ are finite dimensional C^* -algebras. The natural embeddings $\iota_n : \mathcal{F}(\Lambda_n) \rightarrow \mathcal{F}(\Lambda_{n+1})$, which identify the δ and ρ generators, are norm preserving. Using the C^* -inductive limit [16, 22], a C^* -algebra \mathcal{F} , called the field algebra of a G -spin model, can be given by

$$\mathcal{F} = \overline{\bigcup_n \mathcal{F}(\Lambda_n)}.$$

There is an action γ of $D(G)$ on \mathcal{F} in the following. For $x \in \mathbb{Z}$, $l \in \mathbb{Z} + \frac{1}{2}$ and $g, h \in G$, set

$$\begin{aligned}(g, h)\delta_f(x) &= \delta_{g,u}\delta_{hf}(x), \quad \forall f \in G, \\ (g, h)\rho_t(l) &= \delta_{g, hth^{-1}}\rho_g(l), \quad \forall t \in G.\end{aligned}$$

The map γ can be extended for products of generators inductively in the number of generators by the rule

$$(g, h)(fT) = \sum_{(g, h)} (g, h)_{(1)}(f)(g, h)_{(2)}(T),$$

where f is one of the generators in \mathcal{F}_{loc} and T is a finite product of generators. Finally, it is linearly extended both in $D(G)$ and \mathcal{F}_{loc} .

Lemma 2.1. ^[22] The field algebra \mathcal{F} is a $D(G)$ -module algebra with respect to the map γ . Namely, the map γ satisfies the following relations:

$$\begin{aligned}(ab)(T) &= a(b(T)), \\ a(T_1 T_2) &= \sum_{(a)} a_{(1)}(T_1) a_{(2)}(T_2), \\ a(T^*) &= (S(a^*)(T))^*\end{aligned}$$

for $a, b \in D(G)$, $T_1, T_2, T \in \mathcal{F}$.

Set

$$\mathcal{A} = \{F \in \mathcal{F} : a(F) = \varepsilon(a)(F), \quad \forall a \in D(G)\}.$$

We call it an observable algebra in the field algebra \mathcal{F} of G -spin models. Furthermore, one can show that \mathcal{A} is a nonzero C^* -subalgebra of \mathcal{F} , and

$$\mathcal{A} = \{F \in \mathcal{F} : E(F) = F\} \equiv E(\mathcal{F}).$$

Indeed, from the following proposition, one can see that \mathcal{A} is a C^* -subalgebra of \mathcal{F} .

Proposition 2.1. ^[22] The map $E: \mathcal{F} \rightarrow \mathcal{A}$ satisfies the following conditions:

- (1) $E(I) = I$ where I is the unit of \mathcal{F} ;
- (2) (bimodular property) $\forall F_1, F_2 \in \mathcal{A}, F \in \mathcal{F}$,

$$E(F_1 F F_2) = F_1 E(F) F_2;$$

- (3) E is positive.

In the following a linear map Γ from a unital C^* -algebra B onto its unital C^* -subalgebra A with properties (1)-(3) in Proposition 2.1 is called a conditional expectation. If Γ is a conditional expectation from B onto A , then Γ is a projection of norm one [3]. In addition, if $E(Bb) = 0$ implies $b = 0$, for $b \in B$, then we say E is faithful.

2.2 The C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$

This section will give a concrete description of the C^* -basic construction for the inclusion $\mathcal{A} \subset \mathcal{F}$ and some properties about the Jones projection.

Let $\Gamma: B \rightarrow A$ be a faithful conditional expectation. Then B_A (viewing B as a right A -module) is a pre-Hilbert module over A with an A -valued inner product $\langle x, y \rangle = \Gamma(x^*y)$ for $x, y \in B_A$. Let $\overline{B_A}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|\Gamma(x^*x)\|_A^{\frac{1}{2}}, \quad x \in B_A.$$

Then $\overline{B_A}$ is a Hilbert C^* -module over A . Since Γ is faithful, the canonical map $B \rightarrow \overline{B_A}$ is injective. Let $L_A(\overline{B_A})$ be the set of all (right) A -module homomorphisms $T: \overline{B_A} \rightarrow \overline{B_A}$ with an adjoint A -module homomorphism $T^*: \overline{B_A} \rightarrow \overline{B_A}$ such that

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle.$$

Then $L_A(\overline{B_A})$ is a C^* -algebra with the operator norm

$$\|T\| = \sup\{\|T\xi\|: \|\xi\| = 1\}.$$

There is an injective $*$ -homomorphism $\lambda: B \rightarrow L_A(\overline{B_A})$ defined by $\lambda(b)x = bx$ for $x \in B_A$ and $b \in B$, so that B can be viewed as a C^* -subalgebra of $L_A(\overline{B_A})$. Note that the map $\gamma_A: B_A \rightarrow B_A$ defined by $\gamma_A(x) = \Gamma(x)$ for $x \in B_A$ is bounded and thus it can be extended to a bounded linear operator on $\overline{B_A}$, denoted by γ_A again. Then $\gamma_A \in L_A(\overline{B_A})$ and $\gamma_A = \gamma_A^2 = \gamma_A^*$; that is, γ_A is a projection in $L_A(\overline{B_A})$. From now on we call γ_A the Jones projection of Γ . The (reduced) C^* -basic construction is a C^* -subalgebra of $L_A(\overline{B_A})$ defined to be

$$\langle B, \gamma_A \rangle_{C^*} = \overline{\text{span}\{\lambda(x)\gamma_A\lambda(y) \in L_A(\overline{B_A}): x, y \in B\}}^{\|\cdot\|}.$$

For the conditional expectation $E: \mathcal{F} \rightarrow \mathcal{A}$, we shall consider the C^* -basic construction $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$, which is a C^* -subalgebra of $L_{\mathcal{A}}(\overline{\mathcal{F}})$ linearly generated by $\{\lambda(x)e_{\mathcal{A}}\lambda(y): x, y \in \mathcal{F}\}$, where $\overline{\mathcal{F}}$ is the completion of $\mathcal{F}_{\mathcal{A}}$ with respect to the norm $\|x\|_{\mathcal{F}_{\mathcal{A}}} = \|E(x^*x)\|_{\mathcal{A}}^{\frac{1}{2}}$ and $e_{\mathcal{A}}$ is the Jones projection of E .

In order to describe the concrete construction of $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$, we first consider the local C^* -basic construction $\langle \mathcal{F}(\Lambda_{\frac{1}{2}, 2}), e_{\mathcal{A}} \rangle_{C^*}$ (since $E: \mathcal{F}(\Lambda_{\frac{1}{2}, 2}) \rightarrow \mathcal{A}(\Lambda_{1, \frac{3}{2}})$ is also a conditional expectation by Proposition 3.1 in [22]), where $\Lambda_{\frac{1}{2}, 2} = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ and $\mathcal{F}(\Lambda_{\frac{1}{2}, 2})$ is a C^* -subalgebra of the field algebra \mathcal{F} generated by

$$\left\{\delta_g(x), \rho_h(l): g, h \in G, x = 1, 2, l = \frac{1}{2}, \frac{3}{2}\right\}.$$

Also the dimension of $\mathcal{F}(\Lambda_{\frac{1}{2}, 2})$ is finite, then the C^* -algebra $\langle \mathcal{F}(\Lambda_{\frac{1}{2}, 2}), e_{\mathcal{A}} \rangle_{C^*}$ is generated by

$$\left\{\delta_g(x), \rho_h(l), e_{\mathcal{A}} \in L_{\mathcal{A}}(\mathcal{F}(\Lambda_{\frac{1}{2}, 2})): g, h \in G, x = 1, 2, l = \frac{1}{2}, \frac{3}{2}\right\}.$$

By Definition 2.2 and $e_{\mathcal{A}}$ being a projection, we know the linear basis of $\langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ is

$$\left\{ \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2}) e_{\mathcal{A}} \delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2}) : g_i, h_i, s_i, t_i \in G, \text{ for } i = 1, 2 \right\}.$$

We will give some properties about the elements in $L_{\mathcal{A}}(\mathcal{F}(\Lambda_{\frac{1}{2},2}))$ as follows.

Lemma 2.2. (1) As operators on $\mathcal{F}(\Lambda_{\frac{1}{2},2})$, we have $e_{\mathcal{A}} T e_{\mathcal{A}} = E(T) e_{\mathcal{A}}$.

(2) Let $T \in \mathcal{F}(\Lambda_{\frac{1}{2},2})$, then $T \in \mathcal{A}(\Lambda_{1,\frac{3}{2}})$ if and only if $e_{\mathcal{A}} T = T e_{\mathcal{A}}$.

Proof. (1) Suppose that $T = \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})$. It suffices to show that

$$e_{\mathcal{A}} T e_{\mathcal{A}}(F) = E(T) e_{\mathcal{A}}(F)$$

for any $F = \delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2}) \in \mathcal{F}(\Lambda_{\frac{1}{2},2})$.

We can compute

$$\begin{aligned} & e_{\mathcal{A}} T e_{\mathcal{A}}(\delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2})) \\ &= \delta_{t_1 t_2, u} e_{\mathcal{A}} T \left(\frac{1}{|G|} \sum_{f \in G} \delta_{f s_1}(1) \delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}) \right) \\ &= \frac{1}{|G|} \delta_{t_1 t_2, u} E \left(\sum_{f \in G} \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2}) \delta_{f s_1}(1) \delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}) \right) \\ &= \frac{1}{|G|} \delta_{t_1 t_2, u} E \left(\sum_{f \in G} \delta_{g_1}(1) \delta_{h_1 f s_1}(1) \delta_{g_2}(2) \delta_{f s_2}(2) \rho_{h_1 f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1} h_2}(\frac{3}{2}) \right) \\ &= \frac{1}{|G|} \delta_{t_1 t_2, u} E(\delta_{g_1}(1) \delta_{g_2}(2) \delta_{h_1^{-1} g_1 s_1^{-1} s_2}(2) \rho_{g_1 s_1^{-1} t_1 s_1 g_1^{-1} h_1}(\frac{1}{2}) \rho_{h_1^{-1} g_1 s_1^{-1} t_1^{-1} s_1 g_1^{-1} h_1 h_2}(\frac{3}{2})) \\ &= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \delta_{h_1 h_2, u} \delta_{g_2, h_1^{-1} g_1 s_1^{-1} s_2} w_{g_1^{-1} g_2}(\frac{3}{2}) v_{g_1^{-1} h_1^{-1} g_1 s_1^{-1} t_1^{-1} s_1}(1), \end{aligned}$$

and

$$\begin{aligned} & E(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) e_{\mathcal{A}}(\delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2})) \\ &= E(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) E(\delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2})) \\ &= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \delta_{h_1 h_2, u} w_{g_1^{-1} g_2}(\frac{3}{2}) v_{g_1^{-1} h_1^{-1} g_1(1)} w_{s_1^{-1} s_2}(\frac{3}{2}) v_{s_1^{-1} t_1^{-1} s_1}(1) \\ &= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \delta_{h_1 h_2, u} \delta_{g_2, h_1^{-1} g_1 s_1^{-1} s_2} w_{g_1^{-1} g_2}(\frac{3}{2}) v_{g_1^{-1} h_1^{-1} g_1 s_1^{-1} t_1^{-1} s_1}(1). \end{aligned}$$

(2) For $T \in \mathcal{A}(\Lambda_{1,\frac{3}{2}})$, without loss of generality, set $T = w_y(\frac{3}{2}) v_x(1)$. Notice that

$$\begin{aligned} & e_{\mathcal{A}} T(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) \\ &= E(w_y(\frac{3}{2}) v_x(1) \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) \\ &= E(\sum_s \delta_s(1) \delta_{s y}(2) \rho_{s x^{-1} s^{-1}}(\frac{1}{2}) \rho_{s x s^{-1}}(\frac{3}{2}) \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) \\ &= E(\sum_s \delta_s(1) \delta_{s x^{-1} s^{-1} g_1}(1) \delta_{s y}(2) \delta_{g_2}(2) \rho_{s x^{-1} s^{-1} h_1}(\frac{1}{2}) \rho_{h_1^{-1} s x s^{-1} h_1 h_2}(\frac{3}{2})) \\ &= E(\delta_{g_1 x^{-1}(1)} \delta_{g_1 x^{-1} y}(2) \delta_{g_2}(2) \rho_{g_1 x^{-1} g^{-1} h_1}(\frac{1}{2}) \rho_{h_1^{-1} g_1 x g^{-1} h_1 h_2}(\frac{3}{2})) \\ &= \frac{1}{|G|} \delta_{g_1 x^{-1} y, g_2} \delta_{h_1 h_2, u} w_{x g_1^{-1} g_2}(\frac{3}{2}) v_{x g_1^{-1} h_1^{-1} g_1}(1), \end{aligned}$$

and

$$\begin{aligned} & T e_{\mathcal{A}}(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})) \\ &= \frac{1}{|G|} \delta_{h_1 h_2, u} w_y(\frac{3}{2}) v_x(1) w_{g_1 g_2^{-1}}(\frac{3}{2}) v_{g_1^{-1} h_1^{-1} g_1}(1) \\ &= \frac{1}{|G|} \delta_{h_1 h_2, u} w_y(\frac{3}{2}) w_{x g_1 g_2^{-1}}(\frac{3}{2}) v_{x g_1^{-1} h_1^{-1} g_1}(1) \\ &= \frac{1}{|G|} \delta_{g_1 x^{-1} y, g_2} \delta_{h_1 h_2, u} w_{x g_1^{-1} g_2}(\frac{3}{2}) v_{x g_1^{-1} h_1^{-1} g_1}(1), \end{aligned}$$

we can conclude that $e_{\mathcal{A}}$ commutes with T if $T \in \mathcal{A}(\Lambda_{1, \frac{3}{2}})$.

For the converse, since $E(I) = I$ where I is the unit of \mathcal{F} , then

$$T = TE(I) = (Te_{\mathcal{A}})(I) = (e_{\mathcal{A}}T)(I) = E(TI) = E(T),$$

which yields that $T \in \mathcal{A}(\Lambda_{1, \frac{3}{2}})$. □

3 C^* -isomorphism between $\mathcal{F} \rtimes D(G)$ and $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$

In this section, we will construct the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$ extending $\mathcal{F} \equiv \mathcal{F} \rtimes I_{D(G)}$ by means of a Hopf module left action of $D(G)$ on \mathcal{F} , such that $\mathcal{F} \rtimes D(G)$ coincides with the C^* -algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

As we have known, the field algebra \mathcal{F} of G -spin models is a $D(G)$ -module algebra, and one can construct the crossed product $*$ -algebra $\mathcal{F}_{\text{loc}} \rtimes D(G)$, as a vector space $\mathcal{F}_{\text{loc}} \otimes D(G)$ with the $*$ -algebra structure

$$\begin{aligned} (T \otimes (g, h)) (F \otimes (s, t)) &= \sum_{(g, h)} T(g, h)_{(1)}(F) \otimes (g, h)_{(2)}(s, t), \\ (T \otimes (g, h))^* &= (I_{\mathcal{F}} \otimes (g, h)^*) (T^* \otimes I_{D(G)}). \end{aligned}$$

The crossed product $\mathcal{F}_{\text{loc}} \rtimes D(G)$ can be extended naturally to a C^* -algebra $\mathcal{F} \rtimes D(G)$ in the following way. Firstly, for any finite subset $\Lambda \subseteq \frac{1}{2}\mathbb{Z}$, let $\mathcal{F}(\Lambda) \rtimes D(G)$ be the subalgebra of $\mathcal{F}_{\text{loc}} \rtimes D(G)$ generated by

$$\left\{ T \otimes (g, h) : T \in \mathcal{F}(\Lambda), (g, h) \in D(G) \right\}.$$

In particular, we consider an increasing sequence of intervals Λ_n for any $n \in \mathbb{N}$, where

$$\begin{aligned} \Lambda_{2n} &= \{s \in \tfrac{1}{2}\mathbb{Z} : -n + \tfrac{1}{2} \leq s \leq n\}, \\ \Lambda_{2n+1} &= \{s \in \tfrac{1}{2}\mathbb{Z} : -n - \tfrac{1}{2} \leq s \leq n\}. \end{aligned}$$

In [4] the authors have shown that smash product $A \# H$ for a finite dimensional Hopf algebra H acting on an algebra A is semisimple if H and A are semisimple. Hence, $\mathcal{F}(\Lambda_n) \rtimes D(G)$ are semisimple. Moreover, $\mathcal{F}(\Lambda_n) \rtimes D(G)$ are finite dimensional C^* -algebras. The natural embeddings $\iota_n : \mathcal{F}(\Lambda_n) \rtimes D(G) \rightarrow \mathcal{F}(\Lambda_{n+1}) \rtimes D(G)$, which identify the T and (g, h) , are norm preserving. Using the C^* -inductive limit [16], a crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$ can be given by

$$\mathcal{F} \rtimes D(G) = \overline{\bigcup_n (\mathcal{F}(\Lambda_n) \rtimes D(G))}.$$

To make further investigation about a C^* -algebra $\mathcal{F} \rtimes D(G)$, we first study a local net structure. For $\Lambda_{\frac{1}{2}, 2} \subset \frac{1}{2}\mathbb{Z}$, we can construct the crossed product C^* -algebra $\mathcal{F}(\Lambda_{\frac{1}{2}, 2}) \rtimes D(G)$.

Using the linear basis $\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \rtimes (g, h) \in \mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$, the multiplication and $*$ -operation are given as follows

$$\begin{aligned}
& \left(\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \rtimes (g, h) \right) \left(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \rtimes (s, t) \right) \\
&= \sum_{(g,h)} \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2})(g, h)_{(1)} (\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})) \rtimes (g, h)_{(2)}(s, t) \\
&= \sum_{f \in G} \delta_{t_1 t_2, h^{-1} f h} \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \delta_{h s_1}(1)\delta_{h s_2}(2)\rho_{h t_1 h^{-1}}(\frac{1}{2})\rho_{h t_2 h^{-1}}(\frac{3}{2}) \\
&= \sum_{f \in G} \delta_{t_1 t_2, h^{-1} f h} \delta_{g_1}(1)\delta_{h_1 h s_1}(1)\delta_{g_2}(2)\delta_{h_1 h_2 h s_2}(2)\rho_{h_1 h t_1 h^{-1}}(\frac{1}{2})\rho_{h t_1^{-1} h^{-1} h_2 h t_1 t_2 h^{-1}}(\frac{3}{2}) \rtimes \delta_{f^{-1} g h, h s}(f^{-1} g, h t) \\
&= \delta_{h^{-1} g h, t_1 t_2 s} \delta_{g_1, h_1 h s_1} \delta_{g_2, h_1 h_2 h s_2} \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1 h t_1 h^{-1}}(\frac{1}{2})\rho_{h t_1^{-1} h^{-1} h_2 h t_1 t_2 h^{-1}}(\frac{3}{2}) \rtimes (h s h^{-1}, h t),
\end{aligned}$$

where the first equation uses the relation

$$\begin{aligned}
& (f, h) \left(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \right) \\
&= \sum_{(f)} f_{(1)} \delta_{s_1}(1) f_{(2)} \delta_{s_2}(2) f_{(3)} \rho_{t_1}(\frac{1}{2}) f_{(4)} \rho_{t_2}(\frac{3}{2}) \\
&= \sum_{x_1, x_2, x_3 \in G} (x_3, h) \delta_{s_1}(1) (x_3^{-1} x_2, h) \delta_{s_2}(2) (x_2^{-1} x_1, h) \rho_{t_1}(\frac{1}{2}) (x_1^{-1} f, h) \rho_{t_2}(\frac{3}{2}) \\
&= \delta_{x_3, u} \delta_{h s_1}(1) \delta_{x_3^{-1} x_2, u} \delta_{h s_2}(2) \delta_{x_2^{-1} x_1, h t_1 h^{-1}} \rho_{h t_1 h^{-1}}(\frac{1}{2}) \delta_{x_1^{-1} f, h t_2 h^{-1}} \rho_{h t_2 h^{-1}}(\frac{3}{2}) \\
&= \delta_{h^{-1} f h, t_1 t_2} \delta_{h s_1}(1) \delta_{h s_2}(2) \rho_{h t_1 h^{-1}}(\frac{1}{2}) \rho_{h t_2 h^{-1}}(\frac{3}{2}),
\end{aligned}$$

and

$$\begin{aligned}
& \left(\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \rtimes (g, h) \right)^* \\
&= \left(\sum_{s_1, s_2 \in G} \delta_{s_1}(1)\delta_{s_2}(2) \rtimes (h^{-1} g h, h^{-1}) \right) \left(\delta_{h_1 g_1^{-1}}(1)\delta_{h_2^{-1} h_1^{-1} g_2}(2)\rho_{h_1^{-1}}(\frac{1}{2})\rho_{h_1 h_2^{-1} h_1^{-1}}(\frac{3}{2}) \rtimes \sum_{s \in G} (s, u) \right) \\
&= \sum_{s_1, s_2, s \in G} \delta_{g, h_2^{-1} h_1^{-1} s} \delta_{s_1, h^{-1} h_1^{-1} g_1} \delta_{s_2, h^{-1} h_2^{-1} h_1^{-1} g_2} \delta_{s_1}(1)\delta_{s_2}(2)\rho_{h^{-1} h_1^{-1} h}(\frac{1}{2})\rho_{h^{-1} h_1 h_2^{-1} h_1^{-1} h}(\frac{3}{2}) \rtimes (h^{-1} s h, h^{-1}) \\
&= \delta_{h^{-1} h_1^{-1} g_1}(1)\delta_{h^{-1} h_2^{-1} h_1^{-1} g_2}(2)\rho_{h^{-1} h_1^{-1} h}(\frac{1}{2})\rho_{h^{-1} h_1 h_2^{-1} h_1^{-1} h}(\frac{3}{2}) \rtimes (h^{-1} h_1 h_2 g h, h^{-1}).
\end{aligned}$$

Now, we consider a special element $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)$ in $\mathcal{F} \rtimes D(G)$.

Lemma 3.1. The element $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)$ is a self-adjoint idempotent element. That is

$$\left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g) \right)^2 = I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g) = \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g) \right)^*.$$

Proof. We can compute that

$$\begin{aligned}
& \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g) \right)^2 \\
&= \left(\sum_{g_1, g_2 \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \frac{1}{|G|} \sum_{h \in G} (u, h) \right) \left(\sum_{s_1, s_2 \in G} \delta_{s_1}(1)\delta_{s_2}(2) \rtimes \frac{1}{|G|} \sum_{t \in G} (u, t) \right) \\
&= \frac{1}{|G|^2} \delta_{g_1, h s_1} \delta_{g_2, h s_2} \sum_{g_1, g_2, s_1, s_2, h, t \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes (u, h t) \\
&= \frac{1}{|G|^2} \sum_{g_1, g_2, h, t \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes (u, h t) \\
&= \frac{1}{|G|} \left(I_{\mathcal{F}} \rtimes \sum_{h \in G} (u, h) \right),
\end{aligned}$$

and

$$\begin{aligned}
\left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right)^* &= \frac{1}{|G|} \left(I_{\mathcal{F}} \rtimes \sum_{g \in G} (u, g)^*\right) \left(I_{\mathcal{F}} \rtimes I_{D(G)}\right) \\
&= \frac{1}{|G|} \left(I_{\mathcal{F}} \rtimes \sum_{g \in G} (u, g^{-1})\right) \left(I_{\mathcal{F}} \rtimes I_{D(G)}\right) \\
&= \frac{1}{|G|} \left(I_{\mathcal{F}} \rtimes \sum_{g \in G} (u, g)\right).
\end{aligned}$$

Hence, $I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)$ is a self-adjoint idempotent element. \square

Lemma 3.2. The element $T \rtimes I_{D(G)}$ in $\mathcal{F} \rtimes D(G)$ satisfies the following covariant relation

$$\left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \left(T \rtimes I_{D(G)}\right) \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) = \left(E(T) \rtimes I_{D(G)}\right) \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right).$$

Proof. Suppose that $T = \delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})$, we can obtain

$$\begin{aligned}
&\left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \left(T \rtimes I_{D(G)}\right) \\
&= \frac{1}{|G|} \left(\sum_{g_1, g_2 \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \sum_{g \in G} (u, g)\right) \left(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \rtimes \sum_{s \in G} (s, u)\right) \\
&= \frac{1}{|G|} \sum_{g_1, g_2, g, s \in G} \delta_{u, t_1 t_2 s} \delta_{g_1, g s_1} \delta_{g_2, g s_2} \delta_{g_1}(1)\delta_{g_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (g s g^{-1}, g) \\
&= \frac{1}{|G|} \sum_{g \in G} \delta_{g s_1}(1)\delta_{g s_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (g t_2^{-1} t_1^{-1} g^{-1}, g),
\end{aligned}$$

and then

$$\begin{aligned}
&\left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \left(T \rtimes I_{D(G)}\right) \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \\
&= \frac{1}{|G|} \sum_{g \in G} \delta_{g s_1}(1)\delta_{g s_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (g t_2^{-1} t_1^{-1} g^{-1}, g) \left(\sum_{g_1, g_2 \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \\
&= \frac{1}{|G|^2} \sum_{g, g_1, g_2, f \in G} \delta_{t_1 t_2, u} \delta_{s_1, t_1 g_1} \delta_{s_2, g_2} \delta_{g s_1}(1)\delta_{g s_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (u, g f) \\
&= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \sum_{g, f \in G} \delta_{g s_1}(1)\delta_{g s_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (u, g f) \\
&= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \sum_{g, f \in G} \delta_{g s_1}(1)\delta_{g s_2}(2) \rho_{g t_1 g^{-1}}(\frac{1}{2}) \rho_{g t_2 g^{-1}}(\frac{3}{2}) \rtimes (u, f).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E(T) &= E\left(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})\right) \\
&= \frac{1}{|G|} \delta_{t_1 t_2, u} \sum_{f \in G} \delta_{f s_1}(1)\delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}),
\end{aligned}$$

and then

$$\begin{aligned}
&\left(E(T) \rtimes I_{D(G)}\right) \left(I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g)\right) \\
&= \left(\frac{1}{|G|} \delta_{t_1 t_2, u} \sum_{f \in G} \delta_{f s_1}(1)\delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}) \rtimes \sum_{s \in G} (s, u)\right) \left(\sum_{g_1, g_2 \in G} \delta_{g_1}(1)\delta_{g_2}(2) \rtimes \frac{1}{|G|} \sum_{t \in G} (u, t)\right) \\
&= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \sum_{f, t, g_1, g_2, s \in G} \delta_{s, u} \delta_{s_1, t_1 f^{-1} g_1} \delta_{f s_2, g_2} \delta_{f s_1}(1)\delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}) \rtimes (u, t) \\
&= \frac{1}{|G|^2} \delta_{t_1 t_2, u} \sum_{f, t \in G} \delta_{f s_1}(1)\delta_{f s_2}(2) \rho_{f t_1 f^{-1}}(\frac{1}{2}) \rho_{f t_2 f^{-1}}(\frac{3}{2}) \rtimes (u, t).
\end{aligned}$$

From the above, we can obtain the desired result. \square

The following theorem is one of main results of this paper, which gives a characterization of the C^* -algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

Theorem 3.1. There exists a C^* -isomorphism between the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$ and the C^* -algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$. That is,

$$\mathcal{F} \rtimes D(G) \cong \langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}.$$

Proof. Let

$$\Phi_{\frac{1}{2},2}: \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*} \rightarrow \mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$$

be a map with

$$\begin{aligned} T &\mapsto T \rtimes I_{D(G)}, \\ e_{\mathcal{A}} &\mapsto I_{\mathcal{F}} \rtimes \frac{1}{|G|} \sum_{g \in G} (u, g). \end{aligned}$$

Firstly, $\Phi_{\frac{1}{2},2}$ is well-defined and can be linearly extended in $\langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ preserving algebraic structure. Actually, by Lemma 2.2 and Lemma 3.2, we obtain that

$$\Phi_{\frac{1}{2},2}(e_{\mathcal{A}} T e_{\mathcal{A}}) = \Phi_{\frac{1}{2},2}(e_{\mathcal{A}}) \Phi_{\frac{1}{2},2}(T) \Phi_{\frac{1}{2},2}(e_{\mathcal{A}}) = \Phi_{\frac{1}{2},2}(E(T) e_{\mathcal{A}}).$$

Also we have for $T_1 = \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})$, $T_2 = \delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2})$,

$$\begin{aligned} T_1 T_2 &= \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2}) \delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2}) \\ &= \delta_{g_1, h_1 s_1} \delta_{g_2, h_1 h_2 s_2} \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1 t_1}(\frac{1}{2}) \rho_{t_1^{-1} h_2 t_1 t_2}(\frac{3}{2}) \end{aligned}$$

and

$$\begin{aligned} &\Phi_{\frac{1}{2},2}(T_1) \Phi_{\frac{1}{2},2}(T_2) \\ &= \left(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2}) \rtimes I_{D(G)} \right) \left(\delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2}) \rtimes I_{D(G)} \right) \\ &= \left(\delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2}) \rtimes \sum_{g \in G} (g, u) \right) \left(\delta_{s_1}(1) \delta_{s_2}(2) \rho_{t_1}(\frac{1}{2}) \rho_{t_2}(\frac{3}{2}) \rtimes \sum_{s \in G} (s, u) \right) \\ &= \sum_{g, s \in G} \delta_{g, t_1 t_2 s} \delta_{g_1, h_1 s_1} \delta_{g_2, h_1 h_2 s_2} \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1 t_1}(\frac{1}{2}) \rho_{t_1^{-1} h_2 t_1 t_2}(\frac{3}{2}) \rtimes (s, u) \\ &= \delta_{g_1, h_1 s_1} \delta_{g_2, h_1 h_2 s_2} \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1 t_1}(\frac{1}{2}) \rho_{t_1^{-1} h_2 t_1 t_2}(\frac{3}{2}) \rtimes I_{D(G)} \\ &= \Phi_{\frac{1}{2},2}(T_1 T_2). \end{aligned}$$

Thus, $\Phi_{\frac{1}{2},2}$ is an algebra homomorphism.

Secondly, $\Phi_{\frac{1}{2},2}$ preserves the $*$ -operation. In fact, we can show for $T = \delta_{g_1}(1) \delta_{g_2}(2) \rho_{h_1}(\frac{1}{2}) \rho_{h_2}(\frac{3}{2})$, $T^* = \delta_{h_1^{-1} g_1}(1) \delta_{h_2^{-1} h_1^{-1} g_2}(2) \rho_{h_1^{-1}}(\frac{1}{2}) \rho_{h_1 h_2^{-1} h_1^{-1}}(\frac{3}{2})$ and

$$\begin{aligned} \left(\Phi_{\frac{1}{2},2}(T) \right)^* &= \sum_{g \in G} \delta_{h_1^{-1} g_1}(1) \delta_{h_2^{-1} h_1^{-1} g_2}(2) \rho_{h_1^{-1}}(\frac{1}{2}) \rho_{h_1 h_2^{-1} h_1^{-1}}(\frac{3}{2}) \rtimes (h_1 h_2 g, u) \\ &= T^* \rtimes I_{D(G)} \\ &= \Phi_{\frac{1}{2},2}(T^*). \end{aligned}$$

It follows from Lemma 3.1 that $\Phi_{\frac{1}{2},2}(e_{\mathcal{A}})$ is self-adjoint, which together with $e_{\mathcal{A}}$ is a projection yields that $(\Phi_{\frac{1}{2},2}(e_{\mathcal{A}}))^* = \Phi_{\frac{1}{2},2}(e_{\mathcal{A}}) = \Phi_{\frac{1}{2},2}(e_{\mathcal{A}}^*)$.

In order to complete the proof we need to show that $\Phi_{\frac{1}{2},2}$ is bijective. Indeed, For any $\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \rtimes (g, h) \in \mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$, choose

$\left(|G|\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2g}(\frac{3}{2})\right)e_{\mathcal{A}}\left(\delta_{h^{-1}h_1^{-1}g_1}(1)\delta_{h^{-1}g^{-1}h_2^{-1}h_1^{-1}g_2}(2)\rho_u(\frac{1}{2})\rho_{h^{-1}g^{-1}h}(\frac{3}{2})\right) \in \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ such that

$$\begin{aligned} & \Phi_{\frac{1}{2},2}\left(|G|\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2g}(\frac{3}{2})\right)e_{\mathcal{A}}\left(\delta_{h^{-1}h_1^{-1}g_1}(1)\delta_{h^{-1}g^{-1}h_2^{-1}h_1^{-1}g_2}(2)\rho_u(\frac{1}{2})\rho_{h^{-1}g^{-1}h}(\frac{3}{2})\right) \\ &= \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}) \rtimes (g, h), \end{aligned}$$

which implies that $\Phi_{\frac{1}{2},2}$ is surjective, and then $\dim \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*} \geq |G|^6$. Also we can show that every element

$$\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2})e_{\mathcal{A}}(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})) \in \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*},$$

as a operator on $\mathcal{F}(\Lambda_{\frac{1}{2},2})$, is equal to the element

$$\delta_{f_1}(1)\delta_{f_2}(2)\rho_{\varphi_1}(\frac{1}{2})\rho_{\varphi_2g}(\frac{3}{2})e_{\mathcal{A}}(\delta_{h^{-1}\varphi_1^{-1}f_1}(1)\delta_{h^{-1}g^{-1}\varphi_2^{-1}\varphi_1^{-1}f_2}(2)\rho_u(\frac{1}{2})\rho_{h^{-1}g^{-1}h}(\frac{3}{2})) \in \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*},$$

where

$$\begin{aligned} f_1 &= g_1, & f_2 &= g_2, \\ \varphi_1 &= g_1s_1^{-1}t_1s_1g_1^{-1}h_1, \\ \varphi_2 &= h_1^{-1}g_1s_1^{-1}t_1^{-1}s_1g_1^{-1}h_1h_2h_1^{-1}g_1s_1^{-1}t_1t_2s_1g_1^{-1}h_1, \\ g &= h_1^{-1}g_1s_1^{-1}t_2^{-1}t_1^{-1}s_1g_1^{-1}h_1, \\ h &= h_1^{-1}g_1s_1^{-1}. \end{aligned}$$

Hence, $\Phi_{\frac{1}{2},2}$ is a C^* -homomorphism and is bijective. By Theorem 2.1.7 in [18], $\Phi_{\frac{1}{2},2}$ is a C^* -isomorphism from $\langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ onto $\mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$.

By induction, we can build a C^* -isomorphism $\Phi_{\frac{n}{2},m}$ between $\langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*}$ and $\mathcal{F}(\Lambda_{\frac{n}{2},m}) \rtimes D(G)$ for any $n, m \in \mathbb{Z}$ with $n < m$.

Now, because of the last relation we can define a C^* -isomorphism

$$\Phi: \bigcup_{n < m} \langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*} \rightarrow \bigcup_{n < m} (\mathcal{F}(\Lambda_{\frac{n}{2},m}) \rtimes D(G))$$

by $\Phi|_{\langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*}} = \Phi_{\frac{n}{2},m}$. Since each $\Phi_{\frac{n}{2},m}$ is an isometry, Φ is an isometry between $\bigcup_{n < m} \langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*}$ and $\bigcup_{n < m} (\mathcal{F}(\Lambda_{\frac{n}{2},m}) \rtimes D(G))$. Then by Theorem 2.7 in [2], The map Φ can be extended to an isomorphism from $\overline{\bigcup_{n < m} \langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*}}$ onto $\overline{\bigcup_{n < m} (\mathcal{F}(\Lambda_{\frac{n}{2},m}) \rtimes D(G))}$.

Finally, the uniqueness of the C^* -inductive limit [16] implies that $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*} = \overline{\bigcup_{n < m} \langle \mathcal{F}(\Lambda_{\frac{n}{2},m}), e_{\mathcal{A}} \rangle_{C^*}}$ and $\mathcal{F} \rtimes D(G) = \overline{\bigcup_{n < m} (\mathcal{F}(\Lambda_{\frac{n}{2},m}) \rtimes D(G))}$. Consequently, the C^* -algebra $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$ is C^* -isomorphic to the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$. \square

Remark 3.1. From Theorem 3.1, we know that the C^* -basic constructions do not depend on the choice of conditional expectations, which can also be seen in Proposition 2.10.11 [24].

4 The conditional expectation from $\mathcal{F} \rtimes D(G)$ onto \mathcal{F}

Let $H(m, \iota, S, \Delta, \varepsilon)$ be a Hopf C^* -algebra with finite dimension, where $m, \iota, S, \Delta, \varepsilon$ denote multiplication, unit, antipode, comultiplication and counit, respectively. The dual \widehat{H} of H is also a finite Hopf C^* -algebra with multiplication \widetilde{m} , unit $\widetilde{\iota}$, antipode \widetilde{S} , comultiplication $\widetilde{\Delta}$, and counit $\widetilde{\varepsilon}$ defined by

$$\begin{aligned}\langle \widetilde{\Delta}(\varphi), x \otimes y \rangle &:= \langle \varphi, m(x \otimes y) \rangle, \\ \langle \widetilde{m}(\varphi \otimes \phi), x \rangle &:= \langle \varphi \otimes \phi, \Delta(x) \rangle, \\ \langle \varphi^*, x \rangle &:= \overline{\langle \varphi, S(x)^* \rangle}, \\ \widetilde{\varepsilon}(\varphi) &:= \langle \varphi, 1_H \rangle, \\ \langle 1_{\widehat{H}}, x \rangle &:= \varepsilon(x), \\ \langle \widetilde{S}(\varphi), x \rangle &:= \langle \varphi, S(x) \rangle.\end{aligned}$$

Since $D(G)$ is of finite dimension and $\widehat{C(\overline{G}) \otimes \mathbb{C}G} \cong \mathbb{C}G \otimes C(G)$ as algebras, $\{(y, \delta_x) : y, x \in G\}$ can be viewed as a linear basis of $\widehat{D(G)}$. As the above states, the structure maps on $\widehat{D(G)}$ are the following

$$\begin{aligned}\widetilde{\Delta}(y, \delta_x) &= \sum_{t \in G} (y, \delta_{t^{-1}}) \otimes (tyt^{-1}, \delta_{tx}), & (\text{coproduct}) \\ (y, \delta_x)(w, \delta_z) &= \delta_{x,z}(yw, \delta_x), & (\text{multiplication}) \\ (y, \delta_x)^* &= (y^{-1}, \delta_x), & (*\text{-operation}) \\ \widetilde{\varepsilon}(y, \delta_x) &= \delta_{x,u}, & (\text{counit}) \\ \widetilde{S}(y, \delta_x) &= (x^{-1}y^{-1}x, \delta_{x^{-1}}). & (\text{antipode})\end{aligned}$$

It is easy to see that $I_{\widehat{D(G)}} = \sum_{x \in G} (u, \delta_x)$, and there is a unique element $E_2 = \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)$, called an integral element, satisfying for any $b \in \widehat{D(G)}$,

$$bE_2 = E_2b = \varepsilon(b)E_2.$$

The map $\sigma : \widehat{D(G)} \times (\mathcal{F} \rtimes D(G)) \rightarrow \mathcal{F} \rtimes D(G)$ given on the generating elements of $\mathcal{F} \rtimes D(G)$ as

$$\sigma((y, \delta_x) \times (F \otimes (g, h))) = \delta_{x,h}(F \otimes (gy^{-1}, h))$$

for $(g, h) \in D(G)$, can be linearly extended both in $\widehat{D(G)}$ and $\mathcal{F} \rtimes D(G)$. Here and from now on, by $(y, \delta_x)(F \otimes (g, h))$ we always denote $\sigma((y, \delta_x) \times (F \otimes (g, h)))$ for notational convenience.

In particular, considering the action of E_2 on $\mathcal{F} \rtimes D(G)$, we can obtain that

$$E_2(F \otimes (g, h)) = \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(F \otimes (g, h)) = \frac{1}{|G|} \sum_{y \in G} \delta_{h,u}(F \otimes (gy^{-1}, h)) = \frac{1}{|G|} \delta_{h,u}(F \otimes I_{D(G)}),$$

which means that the range of E_2 on $\mathcal{F} \rtimes D(G)$ is contained in \mathcal{F} . Moreover, we can show that E_2 is a positive map preserving the unit and possessing the bimodular property. Namely,

Proposition 4.1. The map $E_2: \mathcal{F} \rtimes D(G) \rightarrow \mathcal{F}$ is a conditional expectation.

Proof. (1) $E_2(I_{\mathcal{F} \rtimes D(G)}) = E_2(I_{\mathcal{F}} \otimes \sum_{g \in G} (g, u)) = \frac{1}{|G|} \sum_{g \in G} (I_{\mathcal{F}} \otimes I_{D(G)}) = I_{\mathcal{F}}$.

(2) $\forall T_1, T_2 \in \mathcal{F}, \tilde{T} \in \mathcal{F} \rtimes D(G)$, we have

$$\begin{aligned}
E_2(T_1 \tilde{T} T_2) &= \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(T_1 \tilde{T} T_2) \\
&= \frac{1}{|G|} \sum_{y, t_1, t_2 \in G} (y, \delta_{t_1^{-1}})(T_1)(t_1 y t_1^{-1}, \delta_{t_1 t_2^{-1}})(\tilde{T})(t_2 y t_2^{-1}, \delta_{t_2})(T_2) \\
&= \frac{1}{|G|} \sum_{y, t_1, t_2 \in G} \tilde{\varepsilon}(y, \delta_{t_1^{-1}})(T_1)(t_1 y t_1^{-1}, \delta_{t_1 t_2^{-1}})(\tilde{T}) \tilde{\varepsilon}(t_2 y t_2^{-1}, \delta_{t_2})(T_2) \\
&= \frac{1}{|G|} \sum_{y, t_1, t_2 \in G} \delta_{t_1^{-1}, u}(T_1)(t_1 y t_1^{-1}, \delta_{t_1 t_2^{-1}})(\tilde{T}) \delta_{t_2, u}(T_2) \\
&= \frac{1}{|G|} \sum_{y \in G} T_1(y, \delta_u)(\tilde{T}) T_2 \\
&= T_1 E_2(\tilde{T}) T_2.
\end{aligned}$$

(3) We note the relation for any $\tilde{T} \in \mathcal{F} \rtimes D(G)$,

$$\begin{aligned}
E_2(\tilde{T}^* \tilde{T}) &= \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(\tilde{T}^* \tilde{T}) \\
&= \frac{1}{|G|} \sum_{y, t \in G} (y, \delta_{t^{-1}})(\tilde{T}^*)(t y t^{-1}, \delta_t)(\tilde{T}) \\
&= \frac{1}{|G|} \sum_{y, t \in G} (\tilde{S}(y, \delta_{t^{-1}})^*(\tilde{T}))^*(t y t^{-1}, \delta_t)(\tilde{T}) \\
&= \frac{1}{|G|} \sum_{y, t \in G} ((t y t^{-1}, \delta_t)(F))^*(t y t^{-1}, \delta_t)(\tilde{T}),
\end{aligned}$$

which means E_2 is a positive map on $\mathcal{F} \rtimes D(G)$. □

We recall some basic facts about the index for C^* -algebras in [24].

Definition 4.1. Let Γ be a conditional expectation from a unital C^* -algebra B onto its unital C^* -subalgebra A . A finite family $\{(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $b \in B$,

$$\sum_{i=1}^n u_i \Gamma(v_i b) = b = \sum_{i=1}^n \Gamma(b u_i) v_i.$$

Furthermore, if there exists a quasi-basis for Γ , we call Γ of index-finite type. In this case we define the index of Γ by

$$\text{Index } \Gamma = \sum_{i=1}^n u_i v_i.$$

Remark 4.1. (1) If Γ is a conditional expectation of index-finite type, then the C^* -index $\text{Index } \Gamma$ is a central element of B and does not depend on the choice of quasi-basis. In particular, if $A \subset B$ are simple unital C^* -algebras, then $\text{Index } \Gamma$ is a positive scalar. Moreover, we can choose one of the form $\{(w_i, w_i^*): i = 1, 2, \dots, n\}$, which shows that $\text{Index } \Gamma$ is a positive element [24].

(2) Let $N \subseteq M$ be factors of type II_1 and $\Gamma: M \rightarrow N$ the canonical conditional expectation determined by the unique normalized trace on M , then $\text{Index } \Gamma$ is exactly Jones index $[M, N]$ based

on the coupling constant [19]. More generally, let M be a (σ -finite) factor with a subfactor N and Γ a normal conditional expectation from M onto N , then Γ is of index-finite if and only if $\text{Index } \Gamma$ is finite in the sense of Ref. [15], and the values of $\text{Index } \Gamma$ are equal.

Next, we consider the conditional expectation from $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ onto the C^* -subalgebra \mathcal{F} with the common unit, where $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ is the C^* -algebra constructed from the C^* -basic construction for the inclusion $\mathcal{A} \subseteq \mathcal{F}$.

Theorem 4.1. The map $\tilde{E}: \langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*} \rightarrow \mathcal{F}$ defined on the basis elements of $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$ by

$$\tilde{E}(Te_{\mathcal{A}}F) = \frac{1}{|\Gamma|^2}TF,$$

linearly extended in $\langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*}$, is a conditional expectation, and we call it the dual conditional expectation of $E: \mathcal{F} \rightarrow \mathcal{A}$. More precisely, set

$$u_{x,y} = |G|^{\frac{3}{2}}\delta_x(k)\rho_y(k + \frac{1}{2})e_{\mathcal{A}}$$

for any $k \in \mathbb{Z}$, then $\{(u_{x,y}, u_{x,y}^*): x, y \in G\}$ is a quasi-basis for \tilde{E} .

Proof. Without loss of generality, one can consider the case of $k = 1$.

Firstly, one can show that $\{(u_{x,y}, u_{x,y}^*): x, y \in G\}$ is a quasi-basis of $\tilde{E}: \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*} \rightarrow \mathcal{F}(\Lambda_{\frac{1}{2},2})$. By hypothesis, we have

$$\begin{aligned} & \tilde{E}\left((\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2}))e_{\mathcal{A}}(\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}))\right) \\ &= \frac{1}{|\Gamma|^2}\delta_{g_1,h_1s_1}\delta_{g_2,h_1h_2s_2}\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1t_1}(\frac{1}{2})\rho_{t_1^{-1}h_2t_1t_2}(\frac{3}{2}). \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{x,y \in G} u_{x,y} \tilde{E}\left[u_{x,y}^* \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})\right] \\ &= |G|^{\frac{3}{2}} \sum_{x,y \in G} u_{x,y} \tilde{E}\left[e_{\mathcal{A}}\delta_x(1)\rho_{y^{-1}}(\frac{3}{2})\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})\right] \\ &= |G|^{\frac{3}{2}} \sum_{x,y \in G} u_{x,y} \tilde{E}\left[e_{\mathcal{A}}\delta_x(1)\delta_{g_1}(1)\delta_{y^{-1}g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_1^{-1}y^{-1}h_1h_2}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2})\right] \\ &= |G| \sum_{y \in G} \delta_{g_1}(1)\rho_y(\frac{3}{2})e_{\mathcal{A}}E(\delta_{g_1}(1)\delta_{y^{-1}g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_1^{-1}y^{-1}h_1h_2}(\frac{3}{2}))\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \\ &= \sum_{y,f \in G} \delta_{y,h_1h_2}\delta_{g_1}(1)\rho_y(\frac{3}{2})\delta_{fg_1}(1)\delta_{fy^{-1}g_2}(2)\rho_{fh_1f^{-1}}(\frac{1}{2})\rho_{fh_1^{-1}f^{-1}}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \\ &= \delta_{g_1}(1)\rho_{h_1h_2}(\frac{3}{2})\delta_{h_2^{-1}h_1^{-1}g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_1^{-1}}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}) \\ &= \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\frac{1}{2})\rho_{h_2}(\frac{3}{2})e_{\mathcal{A}}\delta_{s_1}(1)\delta_{s_2}(2)\rho_{t_1}(\frac{1}{2})\rho_{t_2}(\frac{3}{2}), \end{aligned}$$

which yields that for any $a \in \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$,

$$\sum_{x,y \in G} u_{x,y} \tilde{E}(u_{x,y}^* a) = a.$$

Similarly, one can verify

$$\sum_{x,y \in G} \tilde{E}(au_{x,y})u_{x,y}^* = a, \quad \forall a \in \langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}.$$

By induction, we can show that $\{(u_{x,y}, u_{x,y}^*) : x, y \in G\}$ is a quasi-basis of $\tilde{E} : \langle \mathcal{F}(\Lambda_{n-\frac{1}{2},m}), e_{\mathcal{A}} \rangle_{C^*} \rightarrow \mathcal{F}(\Lambda_{n-\frac{1}{2},m})$ for any $n, m \in \mathbb{Z}$ and $n < m$.

Finally, by the continuity of the conditional expectation \tilde{E} and the uniqueness of the C^* -inductive limit [16], we conclude that $\{(u_{x,y}, u_{x,y}^*) : x, y \in G\}$ is a quasi-basis for $\tilde{E} : \langle \mathcal{F}, e_{\mathcal{A}} \rangle_{C^*} \rightarrow \mathcal{F}$. \square

Remark 4.2. E_2 is consistent with the dual conditional expectation \tilde{E} of $E : \mathcal{F} \rightarrow \mathcal{A}$ in Theorem 3.2. Indeed, it suffices to show that the restriction of E_2 on $\mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$ is consistent with that of \tilde{E} on $\langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ because of the continuity of conditional expectations and the uniqueness of the C^* -inductive limit. Again, it follows from Theorem 3.1 that $\mathcal{F}(\Lambda_{\frac{1}{2},2}) \rtimes D(G)$ is C^* -isomorphic to $\langle \mathcal{F}(\Lambda_{\frac{1}{2},2}), e_{\mathcal{A}} \rangle_{C^*}$ by the map

$$\begin{aligned} & \delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\tfrac{1}{2})\rho_{h_2}(\tfrac{3}{2}) \rtimes (g, h) \\ \mapsto & |G|\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\tfrac{1}{2})\rho_{h_2g}(\tfrac{3}{2})e_{\mathcal{A}}(\delta_{h^{-1}h_1^{-1}g_1}(1)\delta_{h^{-1}g^{-1}h_2^{-1}h_1^{-1}g_2}(2)\rho_u(\tfrac{1}{2})\rho_{h^{-1}g^{-1}h}(\tfrac{3}{2})). \end{aligned}$$

Thus, we have

$$\begin{aligned} & E_2(\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\tfrac{1}{2})\rho_{h_2}(\tfrac{3}{2}) \rtimes (g, h)) \\ = & \frac{1}{|G|}\delta_{h,u}(\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\tfrac{1}{2})\rho_{h_2}(\tfrac{3}{2}) \otimes I_{D(G)}) \\ = & \tilde{E}(|G|\delta_{g_1}(1)\delta_{g_2}(2)\rho_{h_1}(\tfrac{1}{2})\rho_{h_2g}(\tfrac{3}{2}))e_{\mathcal{A}}(\delta_{h^{-1}h_1^{-1}g_1}(1)\delta_{h^{-1}g^{-1}h_2^{-1}h_1^{-1}g_2}(2)\rho_u(\tfrac{1}{2})\rho_{h^{-1}g^{-1}h}(\tfrac{3}{2})). \end{aligned}$$

5 The C^* -basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$

In this section, we continue to investigate the crossed product C^* -algebra $\mathcal{F} \rtimes D(G)$, and the natural $\widehat{D(G)}$ -module algebra action on $\mathcal{F} \rtimes D(G)$, which gives rise to the iterated crossed product C^* -algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$. The fixed point algebra under this action is given by $\mathcal{F} \equiv \mathcal{F} \rtimes I_D(G)$, which is consistent with the range of the conditional expectation E_2 . We then prove that the C^* -algebra $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$ is precisely C^* -isomorphic to the iterated crossed product C^* -algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$.

Proposition 5.1. The map σ defines a Hopf module left action of $\widehat{D(G)}$ on $\mathcal{F} \rtimes D(G)$. That is $\mathcal{F} \rtimes D(G)$ is a left $\widehat{D(G)}$ -module algebra.

Proof. It suffices to check that the map $\sigma : \widehat{D(G)} \times (\mathcal{F} \rtimes D(G)) \rightarrow \mathcal{F} \rtimes D(G)$ satisfies the following relations:

$$\begin{aligned} & \left((y, \delta_x)(w, \delta_z) \right) \left(F \otimes (g, h) \right) = (y, \delta_x) \left((w, \delta_z)(F \otimes (g, h)) \right), \\ & (y, \delta_x) \left((F \otimes (g_1, h_1))(T \otimes (g_2, h_2)) \right) = \sum_{(y, \delta_x)} \left((y, \delta_x)_{(1)}(F \otimes (g_1, h_1)) \right) \left((y, \delta_x)_{(2)}(T \otimes (g_2, h_2)) \right), \\ & (y, \delta_x) \left(F \otimes (g, h) \right)^* = \left(\tilde{S}(y, \delta_x)^*(F \otimes (g, h)) \right)^*, \end{aligned}$$

for $(y, \delta_x), (w, \delta_z) \in \widehat{D(G)}$, $T, F \in \mathcal{F}$ and $(g_i, h_i), (g, h) \in D(G)$ for $i = 1, 2$.

As to the first equality, we compute

$$\begin{aligned}
\left((y, \delta_x)(w, \delta_z)\right)(F \otimes (g, h)) &= \delta_{x,z}(yw, \delta_x)(F \otimes (g, h)) \\
&= \delta_{x,z}\delta_{x,h}(F \otimes (gw^{-1}y^{-1}, h)) \\
&= \delta_{z,h}\delta_{x,h}(F \otimes (gw^{-1}y^{-1}, h)) \\
&= \delta_{z,h}(y, \delta_x)(F \otimes (gw^{-1}, h)) \\
&= (y, \delta_x)\left((w, \delta_z)(F \otimes (g, h))\right).
\end{aligned}$$

Next,

$$\begin{aligned}
&(y, \delta_x)\left((F \otimes (g_1, h_1))(T \otimes (g_2, h_2))\right) \\
&= \sum_{(g_1, h_1)} (y, \delta_x)\left(F(g_1, h_1)_{(1)}T \otimes (g_1, h_1)_{(2)}(g_2, h_2)\right) \\
&= \sum_{f \in G} (y, \delta_x)\left(F(f, h_1)T \otimes (f^{-1}g_1, h_1)(g_2, h_2)\right) \\
&= \sum_{f \in G} (y, \delta_x)\left(F(f, h_1)T \otimes \delta_{f^{-1}g_1, h_1, h_1g_2}(f^{-1}g_1, h_1h_2)\right) \\
&= (y, \delta_x)\left(F(g_1h_1g_2^{-1}h_1^{-1}, h_1)T \otimes (h_1g_2h_1^{-1}, h_1h_2)\right) \\
&= \delta_{x, h_1h_2}F(g_1h_1g_2^{-1}h_1^{-1}, h_1)T \otimes (h_1g_2h_1^{-1}y^{-1}, h_1h_2),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{(y, \delta_x)} \left((y, \delta_x)_{(1)}(F \otimes (g_1, h_1))\right) \left((y, \delta_x)_{(2)}(T \otimes (g_2, h_2))\right) \\
&= \sum_{t \in G} \left((y, \delta_{t^{-1}})(F \otimes (g_1, h_1))\right) \left((tyt^{-1}, \delta_{tx})(T \otimes (g_2, h_2))\right) \\
&= \sum_{t \in G} \delta_{t^{-1}, h_1} \delta_{tx, h_2} \left(F \otimes (g_1y^{-1}, h_1)\right) \left(T \otimes (g_2ty^{-1}t^{-1}, h_2)\right) \\
&= \delta_{x, h_1h_2} \left(F \otimes (g_1y^{-1}, h_1)\right) \left(T \otimes (g_2h_1^{-1}y^{-1}h_1, h_2)\right) \\
&= \sum_{f \in G} \delta_{x, h_1h_2} F(f, h_1)T \otimes (f^{-1}g_1y^{-1}, h_1)(g_2h_1^{-1}y^{-1}h_1, h_2) \\
&= \delta_{x, h_1h_2} \delta_{f^{-1}g_1, h_1g_2h_1^{-1}} F(f, h_1)T \otimes (f^{-1}g_1y^{-1}, h_1h_2) \\
&= \delta_{x, h_1h_2} F(g_1h_1g_2^{-1}h_1^{-1}, h_1)T \otimes (h_1g_2h_1^{-1}y^{-1}, h_1h_2).
\end{aligned}$$

Thus, we obtain that

$$(y, \delta_x)\left((F \otimes (g_1, h_1))(T \otimes (g_2, h_2))\right) = \sum_{(y, \delta_x)} \left((y, \delta_x)_{(1)}(F \otimes (g_1, h_1))\right) \left((y, \delta_x)_{(2)}(T \otimes (g_2, h_2))\right).$$

To prove the third equation, we can calculate

$$\begin{aligned}
(y, \delta_x)\left(F \otimes (g, h)\right)^* &= (y, \delta_x)\left((I_{\mathcal{F}} \otimes (g, h)^*)(F^* \otimes I_{D(G)})\right) \\
&= (y, \delta_x)\left((I_{\mathcal{F}} \otimes (h^{-1}gh, h^{-1}))(F^* \otimes I_{D(G)})\right) \\
&= \sum_{f \in G} (y, \delta_x)\left((f, h^{-1})F^* \otimes (f^{-1}h^{-1}gh, h^{-1})\right) \\
&= \sum_{f \in G} \delta_{x, h^{-1}}(f, h^{-1})F^* \otimes (f^{-1}h^{-1}ghy^{-1}, h^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
\left(\tilde{S}(y, \delta_x)(F \otimes (g, h))\right)^* &= \left((x^{-1}yx, \delta_{x^{-1}})(F \otimes (g, h))\right)^* \\
&= \delta_{x^{-1}, h} \left(F \otimes (gx^{-1}y^{-1}x, h)\right)^* \\
&= \sum_{f \in G} \delta_{x, h^{-1}}(f, h^{-1}) F^* \otimes (f^{-1}h^{-1}ghy^{-1}, h^{-1}).
\end{aligned}$$

□

From Proposition 5.1, we can construct the crossed product $(\mathcal{F} \rtimes D(G)) \rtimes \widehat{D(G)}$, which is called the iterated crossed product C^* -algebra.

In the following, we will consider the $\widehat{D(G)}$ -invariant subalgebra of $\mathcal{F} \rtimes D(G)$. To do this, set

$$(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} = \{\tilde{T} \in \mathcal{F} \rtimes D(G) : b(\tilde{T}) = \tilde{\varepsilon}(b)(\tilde{T}), \quad \forall b \in \widehat{D(G)}\}.$$

One can show that $(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$ is a C^* -subalgebra of $\mathcal{F} \rtimes D(G)$. Furthermore,

$$(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} = \{\tilde{T} \in \mathcal{F} \rtimes D(G) : E_2(\tilde{T}) = \tilde{T}\}.$$

In fact, for $\tilde{T} \in (\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} \subseteq \mathcal{F} \rtimes D(G)$, we can compute that

$$E_2(\tilde{T}) = \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(\tilde{T}) = \frac{1}{|G|} \sum_{y \in G} \tilde{\varepsilon}(y, \delta_u)(\tilde{T}) = \frac{1}{|G|} \sum_{y \in G} \tilde{T} = \tilde{T}.$$

For the converse, suppose that $\tilde{T} \in \mathcal{F} \rtimes D(G)$ with $E_2(\tilde{T}) = \tilde{T}$. Then for any $(w, \delta_z) \in \widehat{D(G)}$, we have

$$\begin{aligned}
(w, \delta_z)(\tilde{T}) &= (w, \delta_z)E_2(\tilde{T}) \\
&= (w, \delta_z) \frac{1}{|G|} \sum_{y \in G} (y, \delta_u)(\tilde{T}) \\
&= \frac{1}{|G|} \sum_{y \in G} \delta_{z, u}(wy, \delta_u)(\tilde{T}) \\
&= \tilde{\varepsilon}(w, \delta_z)E_2(\tilde{T}) \\
&= \tilde{\varepsilon}(w, \delta_z)(\tilde{T}).
\end{aligned}$$

This can be linearly extended in $\widehat{D(G)}$. Hence, $\tilde{T} \in (\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$.

Remark 5.1. $(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}}$ is the subalgebra of $\mathcal{F} \rtimes D(G)$ corresponding to the trivial representation $\tilde{\varepsilon}$ of $\widehat{D(G)}$. In fact, $\widehat{D(G)}$ is semisimple and $\mathcal{F} \rtimes D(G)$ is a $\widehat{D(G)}$ -module algebra, then $\mathcal{F} \rtimes D(G)$ is completely reducible, which means $\mathcal{F} \rtimes D(G)$ can be decomposed into a direct sum

$$\mathcal{F} \rtimes D(G) = \bigoplus_{r \in [\widehat{D(G)}]} (\mathcal{F} \rtimes D(G))^r, \quad (\mathcal{F} \rtimes D(G))^r = M^r(\mathcal{F} \rtimes D(G)),$$

where $[\widehat{D(G)}]$ denotes the set of equivalence classes of irreducible representations of $\widehat{D(G)}$, and $\{M^r, r \in [\widehat{D(G)}]\}$ the set of minimal central idempotents in $\widehat{D(G)}$.

Clearly, $(\tilde{\varepsilon}, \mathcal{F} \rtimes D(G))$ is an irreducible representation of $\widehat{D(G)}$. As we have known that the representations of $\widehat{D(G)}$ are in one-to-one correspondence with the $\widehat{D(G)}$ -modules, then

$$(\mathcal{F} \rtimes D(G))^{\tilde{\varepsilon}} \triangleq \{\tilde{F} \in \mathcal{F} \rtimes D(G) : b(\tilde{F}) = \tilde{\varepsilon}(b)\tilde{F}, \forall b \in \widehat{D(G)}\} = E_2(\mathcal{F} \rtimes D(G))$$

is a $\widehat{D(G)}$ -module. As a consequence, $(\mathcal{F} \rtimes D(G))^{\widehat{D(G)}} = (\mathcal{F} \rtimes D(G))^{\tilde{\varepsilon}}$ corresponds to the trivial representation $\tilde{\varepsilon}$, where $M^{\tilde{\varepsilon}}$ is just E_2 .

Naturally, we consider the C^* -algebra $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$ constructed from the C^* -basic construction for the inclusion $\mathcal{F} \subseteq \mathcal{F} \rtimes D(G)$ in the following, where e_2 is the Jones projection of E_2 .

Theorem 5.1. There exists a C^* -isomorphism of C^* -algebras between $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ and $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$. That is,

$$\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \cong \langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}.$$

Proof. The proof is similar to that of Theorem 3.1. Indeed, considering the map Ψ given by

$$\begin{aligned} \Psi: \quad \langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*} &\rightarrow \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \\ T \rtimes (g, h) &\mapsto T \rtimes (g, h) \rtimes I_{\widehat{D(G)}}, \\ e_2 &\mapsto I_{\mathcal{F} \rtimes D(G)} \rtimes \frac{1}{|G|} \sum_{y \in G} (y, \delta_y), \end{aligned}$$

one can show that Ψ is a C^* -isomorphism from $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ onto $\langle \mathcal{F} \rtimes D(G), e_2 \rangle_{C^*}$. \square

Moreover, by Takai duality [23], the iterated crossed product C^* -algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is canonically isomorphic to $M_{|G|^2}(\mathcal{F})$.

Remark 5.2. In the following we will give the concrete construction for $M_{|G|^2}(\mathcal{F})$.

The local field $M_{|G|^2}(\mathcal{F}_{\text{loc}})$ of a $M_{|G|^2}(G)$ -spin model is a $*$ -algebra with a unit $I_{M_{|G|^2}(\mathcal{F})}$ generated by $\left\{ \delta_g(x) \otimes M, \rho_h(l) \otimes N : g, h \in G, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}, M, N \in \text{LB}(M_{|G|^2}) \right\}$ satisfying the following relations

$$\begin{aligned} O_M^g(x) O_N^h(x) &= \delta_{g,h} O_{MN}^g(x), \\ D_M^g(l) D_N^h(l) &= D_{MN}^{gh}(l), \\ \sum_{g \in G} O_I^g(x) &= I_{M_{|G|^2}(\mathcal{F})} = D_I^u(l), \\ O_M^g(x) O_N^h(x') &= O_M^h(x') O_N^g(x), \\ D_M^g(l) O_N^h(x) &= \begin{cases} O_M^{gh}(x) D_N^g(l), & \text{if } l < x, \\ O_M^h(x) D_N^g(l), & \text{if } l > x, \end{cases} \\ D_M^g(l) D_N^h(l') &= \begin{cases} D_M^h(l') D_N^{h^{-1}gh}(l), & \text{if } l > l', \\ D_M^{ghg^{-1}}(l') D_N^g(l), & \text{if } l < l', \end{cases} \\ (O_M^g(x))^* &= O_{M^*}^g(x), \\ (D_N^h(l))^* &= D_{N^*}^{h^{-1}}(l), \end{aligned}$$

for $x, x' \in \mathbb{Z}$, $l, l' \in \mathbb{Z} + \frac{1}{2}$ and $g, h \in G$, where by $O_M^g(x)$, $D_N^h(l)$ and $\text{LB}(M_{|G|^2})$ we denote $\delta_g(x) \otimes M$, $\rho_h(l) \otimes N$ and the linear basis of $M_{|G|^2}(\mathbb{C})$ for convenience, respectively.

For any finite subset $\Lambda \subseteq \frac{1}{2}\mathbb{Z}$, let $M_{|G|^2}(\mathcal{F}(\Lambda))$ be the $*$ -subalgebra of $M_{|G|^2}(\mathcal{F}_{\text{loc}})$ generated by

$$\left\{ O_M^g(x), D_N^h(l) : g, h \in G, x, l \in \Lambda, M, N \in \text{LB}(M_{|G|^2}) \right\}.$$

Similar to the case of the field algebra \mathcal{F} of a G -spin model, one can show that $M_{|G|^2}(\mathcal{F})$ is the C^* -algebra given by the C^* -inductive limit

$$M_{|G|^2}(\mathcal{F}) = \overline{\bigcup_n M_{|G|^2}(\mathcal{F}(\Lambda_n))}.$$

From now on, we call $M_{|G|^2}(\mathcal{F})$ the field algebra of a $M_{|G|^2}(G)$ -spin model, and we call $O_M^g(x)$ and $D_N^h(l)$ order and disorder operators, respectively.

Now one can show that the field algebra $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is $D(G)$ -module algebra. Indeed, the map

$$\tau : D(G) \times (\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}) \rightarrow \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$$

given on the generating elements of $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ as

$$\tau((g, h) \times (\tilde{F} \otimes (y, \delta_x))) = \delta_{h^{-1}gh, x^{-1}yx}(\tilde{F} \otimes (y, \delta_{xh^{-1}}))$$

for any $\tilde{F} \in \mathcal{F} \rtimes D(G)$, can be linearly extended both in $D(G)$ and $\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$.

The observable algebra of $M_{|G|^2}(G)$ -spin models is defined as $(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)})^{D(G)}$. So it is clear that $(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)})^{D(G)} = E_2(\mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}) = \mathcal{F} \rtimes D(G)$.

Remark 5.3. Let $\mathcal{A} \subseteq \mathcal{F}$ be an inclusion of unital C^* -algebras with a conditional expectation $E : \mathcal{F} \rightarrow \mathcal{A}$ of index-finite type [26]. Set $\mathcal{F}_{-1} = \mathcal{A}$, $\mathcal{F}_0 = \mathcal{F}$, and $E_1 = E$, and recall the C^* -basic construction (the C^* -algebra version of the basic construction). We inductively define $e_{k+1} = e_{\mathcal{F}_{k-1}}$ and $\mathcal{F}_{k+1} = \langle \mathcal{F}_k, e_{k+1} \rangle_{C^*}$, the Jones projection and C^* -basic construction applied to $E_{k+1} : \mathcal{F}_k \rightarrow \mathcal{F}_{k-1}$, and take $E_{k+2} : \mathcal{F}_{k+1} \rightarrow \mathcal{F}_k$ to be the dual conditional expectation $E_{\mathcal{F}_k}$ of Definition 2.3.3 in [24]. Then this gives the inclusion tower of iterated basic constructions

$$\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{F} \rtimes D(G) \subseteq \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \subseteq \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \rtimes D(G) \subseteq \dots$$

It follows from Proposition 2.10.11 of [24] that this tower does not depend on the choice of E .

Notice that $\mathcal{F}_2 = \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)}$ is C^* -isomorphic to $M_{|G|^2}(\mathcal{F})$, the field algebra of a $M_{|G|^2}(G)$ -spin model, and $\mathcal{F}_4 = \mathcal{F} \rtimes D(G) \rtimes \widehat{D(G)} \rtimes D(G) \rtimes \widehat{D(G)}$ is C^* -isomorphic to $M_{|G|^4}(\mathcal{F})$, called the field algebra of a $M_{|G|^4}(G)$ -spin model, where the order and disorder operators can be defined similar to those in Remark 5.2.

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